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1976 J. Phys. A: Math. Gen. 9 207

(http://iopscience.iop.org/0305-4470/9/2/003)

J. Phys. A: Math. Gen., Vol. 9, No. 2, 1976. Printed in Great Britain. © 1976

The conformally invariant action-at-a-distance electrodynamics

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Received 28 April 1975

Abstract. A simple six-dimensional form of the conformally invariant action-at-a-distance for electrodynamics is given which in the Minkowsky space yields the conformally invariant tensor $h_{\mu\nu}$. The field equations are also derived.

1. The action principle

For a number of charges e_i of masses m_i the relativistic action, in the action-at-distance formulation,

$$I=-\sum_i m_i\int (\dot{z}_i^2(\tau))^{1/2}\,d\tau+\tfrac{1}{2}\sum_i e_i\sum_{j\neq i} e_j\int \delta[(z_i-z_j)^2]\dot{z}_i\,.\,\dot{z}_j\,d\tau\,d\tau' \tag{1}$$

is not conformally invariant. Here $z_i(\tau)$, $z_j(\tau')$, ... are the world lines of the particles, and τ,τ', ... arbitrary invariant parametrization of the world lines.

The conformally invariant modified action is given by (Ryder 1974, Boulware *et al* 1970)

$$I^{\mathrm{c}}=-\sum_i m_i\int (\dot{z}_i^2(\tau))^{1/2}\,d\tau+\sum_i e_i\sum_{j\neq i} e_j\int \delta[(z_i-z_j)^2]h_{\mu\nu}(z_iz_j)\dot{z}_i^{\mu}\dot{z}_j^{\nu}\,d\tau\,d\tau' \tag{2}$$

where

$$h_{\mu\nu}(z_iz_j)=-\frac{1}{2}(z_i-z_j)^2\frac{\partial}{\partial z_i^{\mu}}\frac{\partial}{\partial z_j^{\nu}}\ln(z_i-z_j)^2. \tag{3}$$

This complicated tensor is so determined as to make the second term in (2) conformally invariant.

The purpose of this paper is to show that the natural simplest scalar action in the six-dimensional formulation of the conformally invariant theories

$$I^{\mathrm{C}}=-\sum_i \overset{0}{m}_i\int (\dot{\eta}^2)^{1/2}\,d\sigma+\sum_i e_i\sum_{j\neq i} e_j\int \delta[(\eta_i-\eta_j)^2]\dot{\eta}_i\,.\,\dot{\eta}_j\,d\sigma_i\,d\sigma_j \tag{4}$$

leads to the factor (3) in the four-dimensional action (2). Here $\eta(\sigma)$ is the world line of the particle and $\overset{0}{m}$ the 'conformally invariant mass' (Barut and Haugen 1972). This new

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form of the action is in agreement with the view (Barut and Haugen 1972, 1973) that the six-dimensional space can be taken to be the true 'physical space'—the fifth and sixth dimensions being related to the scale and the change of scale from point to point—and that the physical laws take their simplest form in the six-dimensional formulation.

2. The six-dimensional world

We recall that under special conformal transformations

$$z'_\mu = \sigma^{-1}(z)(z_\mu + c_\mu z^2), \qquad \sigma(z) = 1 + 2c^\mu z_\mu + c^2 z^2, \tag{5}$$

with $z^2 = z_\mu z^\mu$, we have

$$z'^2 = \sigma^{-1}(z) z^2, \tag{6}$$

but

$$\dot{z}'^2 = \sigma^{-2}(z)\dot{z}'^2 \qquad \text{and} \qquad (z'_1 - z'_2)^2 = \sigma^{-1}(z_1)\sigma^{-1}(z_2)(z_1 - z_2)^2. \tag{7}$$

The mass m, having a dimension, transforms in a z-dependent fashion:

$$m' = \sigma(z) m. \tag{8}$$

Thus the first term of the action (1), the matter part, is conformally invariant *provided* the mass is transformed also according to (8).

The six-dimensional space is introduced by defining

$$\eta^\mu \equiv \kappa z^\mu$$

and

$$\lambda \equiv \kappa z^2. \tag{9}$$

Then the covariant and contravariant components of a six vector are given by the dimensionless quantities

$$\eta^a = (\eta^\mu, l\kappa, \lambda/l), \qquad \eta_a = (\eta_\mu, -\lambda/2l, -\tfrac{1}{2} l\kappa) \tag{10}$$

where l is some fixed length. The metric tensor is

$$g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \tag{11}$$

with the signature: $+-{-}{-}{-}+$. We have then from (9)–(11):

$$\eta^a \eta_a = \eta^\mu \eta_\mu - \kappa\lambda = 0. \tag{12}$$

Thus the physics takes place on the five-dimensional hypercone embedded in the six-space and given by (12).

The special conformal transformations (5) are now linear in the six-dimensional form

$$\begin{aligned} \eta'^\mu &= \eta^\mu + c^\mu \lambda \\ \kappa' &= 2c^\mu \eta_\mu + \kappa + c^2 \lambda \\ \lambda' &= \lambda. \end{aligned} \tag{13}$$

(Elimination of κ and λ here gives back the non-linear law (5).)

3. Conformal invariance

Both terms of the action (4) are clearly conformally invariant, because $\overset{0}{m}$ and the parameters $\sigma_i, \sigma_j, \ldots$ are. The first term leads correctly to the free-particle geodesic on the cone (12) and reduces to the first term of (2) in the Minkowski space where

$$m = \kappa \overset{0}{m}, \qquad \kappa = \text{constant in Minkowski space.}$$

This has been shown in a previous paper and we do not wish to repeat it (Barut and Haugen 1972).

For the second, interaction term of the action (4) we have first, from (9), the identity:

$$(\eta_i - \eta_j)^2 = \kappa_i \kappa_j (z_i - z_j)^2. \tag{14}$$

Further we can deduce that

$$\dot{\eta}_i \,.\, \dot{\eta}_j = \kappa_i \kappa_j \dot{z}_i \,.\, \dot{z}_j - \tfrac{1}{2}\dot{\kappa}_i \dot{\kappa}_j (z_i - z_j)^2 + \kappa_j \dot{\kappa}_i (z_i - z_j) \,.\, \dot{z}_j - \kappa_i \dot{\kappa}_j (z_i - z_j) \,.\, \dot{z}_i. \tag{15}$$

The δ function in the action (4) and the identity (14) imply the constraint

$$\kappa_i \kappa_j (z_i - z_j)^2 = 0 \tag{16}$$

from which we derive the following two identities, by differentiating with respect to σ_i and σ_j, respectively:

$$\begin{aligned} \dot{\kappa}_i &= -2\kappa_i (z_i - z_j) \,.\, \dot{z}_i/(z_i - z_j)^2 \\ \dot{\kappa}_j &= +2\kappa_j (z_i - z_j) \,.\, \dot{z}_j/(z_i - z_j)^2. \end{aligned} \tag{17}$$

Inserting these into (15) we obtain

$$\dot{\eta}_i \,.\, \dot{\eta}_j = \kappa_i \kappa_j \left(g_{\mu\nu} - 2\frac{(z_i - z_j)_\mu (z_i - z_j)_\nu}{(z_i - z_j)^2} \right) \dot{z}_i^\mu \dot{z}_j^\nu. \tag{18}$$

It is easy to see that the expression in the large parentheses is just the factor $h_{\mu\nu}$ in (3). The factor $\kappa_i \kappa_j$ cancels with the same factor coming from $\delta[(\eta_i - \eta_j)^2]$ and consequently the second term of the action (4) becomes

$$\sum_i e_i \sum_{j \neq i} e_j \int \delta[(\eta_i - \eta_j)^2] h_{\mu\nu} \dot{z}_i^\mu \dot{z}_j^\nu \, \mathrm{d}\tau_i \, \mathrm{d}\tau_j \tag{19}$$

ie the standard action (2).

4. Equations of motion

The result of our considerations is the simple action principle (4) in the six-dimensional space. The variational principle gives the following equation of the motion for the ith particle

$$\begin{gathered} \overset{0}{m}_i \left(\frac{\mathrm{d}u_i^a}{\mathrm{d}\sigma} - s_i^a \right) = e_a \kappa_i u_i^b(\sigma) \sum_{j \neq i} e_j \left(u_j^a(\sigma) \frac{\partial}{\partial \eta_i^b} - u_b^j(\sigma) \frac{\partial}{\partial \eta_a^i} \right) \delta[(\eta_i - \eta_j)^2] \\ s_i^a = \frac{1}{\kappa_i} \partial^a \kappa_i; \qquad s_i^a u_{ia} = 0; \qquad u_{ia} \equiv \frac{\partial \eta_{ia}(\sigma)}{\partial \sigma}. \end{gathered} \tag{20}$$

If we now introduce the six-vector potential

$$a_a(\eta_i) = \overset{0}{\kappa}_i \sum_{j\neq i} e_j \int \mathrm{d}\sigma_j\, \delta[(\eta_i - \eta_j)^2] u_a^j(\sigma_j) \tag{21}$$

and hence the corresponding field

$$f_{ab}(\eta_i) = \kappa_i \sum_{j\neq i} e_j \int \mathrm{d}\sigma_j \left(-u_a^j \frac{\partial}{\partial \eta_i^a} + u_b^j(\sigma_j) \frac{\partial}{\partial \eta_a^i} \right) \delta[(\eta_i - \eta_j)^2] \tag{22}$$

we can write (20) as

$$\overset{0}{m} \left(\frac{\mathrm{d}}{\mathrm{d}\sigma} u^a(\sigma) - s^a(\sigma) \right) = e u^b(\sigma) f_b^a(\eta). \tag{23}$$

The corresponding four-vector potential given by (Dirac 1936)

$$A_\mu(x) = \kappa^{-1}(a_\mu - x_\mu a^\kappa) \tag{24}$$

turns out indeed to be equal to

$$A_\mu(z) = \kappa \sum_{j\neq i} e_j \int \delta[(z_i - z_j)^2] h_{\mu\nu} \dot{z}_j^\nu \,\mathrm{d}\sigma_j \tag{25}$$

and does *not* satisfy in general the Lorentz condition $A^\mu,\ \mu = 0$. This should be so, because the Lorentz condition is not conformally invariant (cf Ryder 1974) but we have a conformally invariant theory. The same is true for the six-potential. Indeed we get

$$a^b_{\ ,b} = (\partial_b \kappa) \sum_{j\neq i} \int \mathrm{d}\sigma_j\, \delta[(\eta_i - \eta_j)^2] u_j^b(\sigma_j). \tag{26}$$

Furthermore, the field f_{ab} has only the necessary components (Barut and Haugen 1972) for from (22) we obtain the required condition

$$f_{ab}(\eta_i) \eta_i^b = 0. \tag{27}$$

Equation (27) is a necessary condition in the six-dimensional formulation of electrodynamics and reduces the 15 components of f^{ab} to ten independent quantities, and as there is a fourfold arbitrariness in f^{ab} we have effectively six independent components as in Maxwell theory in the Minkowski space.

References

Barut A O and Haugen R B 1972 *Ann. Phys., NY* **71** 519–41
—— 1973 *Nuovo Cim.* A **18** 491–510, 511–31
Boulware D G, Brown L S and Peccei R D 1970 *Phys. Rev.* D **2** 293–8
Dirac P A M 1936 *Ann. Math., NY* **37** 657
Ryder L H 1974 *J. Phys. A: Math., Nucl. Gen.* **7** 1817–28